a) i) $f'(z_0) = \lim_{h \to 0} \dfrac{f(z_0+h) - f(z_0)}{h}$ exists at each $z_0 \in D$

ii) $f(z_0) = 0$, $f(z_0) = (z - z_0)^n g(z)$ in a neighbourhood of z_0

g is holomorphic, $g(z_0) \neq 0$.

b) i) $\dfrac{1}{(z-1)(z+1)}$

ii) $z(z-1)(z-2)$

iii) $\operatorname{Re} z$

c) $h = h_1 + i h_2$

if $h_2 = 0$: $f'(z_0) = \lim_{h_1 \to 0} \dfrac{f(z_0+h_1) - f(z_0)}{h_1} = \dfrac{\partial f}{\partial x}(z_0) = \dfrac{\partial u}{\partial x}(z_0) + i \dfrac{\partial v}{\partial x}(z_0)$

if $h_1 = 0$: $f'(z_0) = \lim_{h_2 \to 0} \dfrac{f(z_0+ih_2) - f(z_0)}{ih_2} = \dfrac{1}{i} \dfrac{\partial f}{\partial y}(z_0)$

$= -i\left(\dfrac{\partial u}{\partial y}(z_0) + i \dfrac{\partial v}{\partial y}(z_0) \right) = \dfrac{\partial v}{\partial y}(z_0) - i \dfrac{\partial u}{\partial y}(z_0)$

hence $\underbrace{\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} , \quad \dfrac{\partial u}{\partial y} = - \dfrac{\partial v}{\partial x}}_{\text{Cauchy-Riemann equations}}$ at each $z_0 \in D$

d) $z = re^{i\theta}$ $\quad f(z) = \dfrac{r^5 e^{5i\theta}}{r^4} = e^{4i\theta} z$

$f'(z)$ should be $e^{4i\theta}$ for each θ, contradiction

on the axis: $\theta = 0, \dfrac{\pi}{2}, \pi, \dfrac{3}{2}\pi$

$e^{4i\theta} = 1$

hence $f(z) = z$, it satisfies C-R equations because it agrees with a holomorphic function on the axis

2)

a) $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$ $\qquad \cos z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}$ $\qquad \sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}$

b) false

true

false

true

c) Euler formulas: $\cos z = \frac{e^{iz} + e^{-iz}}{2}$

$\sin z = \frac{e^{iz} - e^{-iz}}{2i}$

$$\cos 2z = \frac{e^{2iz} + e^{-2iz}}{2}$$

$$2\cos^2 z - 1 = 2 \cdot \frac{(e^{iz} + e^{-iz})^2}{4} - 1 = \frac{e^{2iz} + e^{-2iz} + 2 - 2}{2} = \cos 2z$$

d) $z = x + iy$: $\quad e^z = e^{x+iy} = e^x \cos y + i e^x \sin y$

$$z^2 = (x^2 - y^2) + i \cdot 2xy$$

$$e^{z^2} = e^{x^2 - y^2} \cos(2xy) + i \cdot e^{x^2 - y^2} \cos(2xy)$$

$$2z = 2x + i \cdot 2y$$

$$e^{2z} = e^{2x} \cos(2y) + i \cdot e^{2x} \sin(2y)$$

$$e^{e^z} = e^{e^x \cos y} \cos(e^x \sin y) + i \cdot e^{e^x \cos y} \sin(e^x \sin y)$$

a) R = radius of convergence (possibly 0 or ∞)

if $\sum a_n z^n$ converges for $|z| < R$

diverges for $|z| > R$

b) i) $a_n = \dfrac{(-2)^n}{n^3}$ $\qquad \sqrt[n]{|a_n|} \to 2 \qquad R = \dfrac{1}{2}$

ii) $a_n = \dfrac{1}{n^n}$ $\qquad \sqrt[n]{|a_n|} = \dfrac{1}{n} \to 0 \qquad R = \infty$

iii) $a_n = 0$ if n is odd and $a_n = 1$ if n is even

$\limsup \sqrt[n]{|a_n|} = 1 \qquad R = 1$

the function is holomorphic on the disc $\{z : |z| < R\}$

c) if $|z| <$ radius of convergence : $\sum a_n z^n$ converges, $a_n z^n \to 0$

if $|z| >$ radius of convergence :

$|z| \cdot \limsup \sqrt[n]{|a_n|} > 1$

there are infinitely many n s.t. $|z| \sqrt[n]{|a_n|} > 1$

$|z|^n \cdot |a_n| > 1$

$|a_n z^n| > 1$

hence $a_n z^n \not\to 0$

$\sqrt[n]{n} \to 1$, hence $\limsup \sqrt[n]{|a_n|} = \limsup \sqrt[n]{n |a_n|}$

$\sum a_n z^n$ and $\sum n a_n z^n$ has the same radius of convergence

$\sum n a_n z^n = z \cdot f'(z)$, $f'(z)$ also has the same radius of conv.

4

(a)

$\Gamma_{\delta,\varepsilon}$ = "keyhole"

δ = width of corridor

ε = radius of circle around z

$\frac{f(w)}{w-z}$ is holomorphic in the interior of $\Gamma_{\delta,\varepsilon}$

$$\int_{\Gamma_{\delta,\varepsilon}} \frac{f(w)}{w-z}\, dw = 0$$

letting $\delta \to 0$: since $\frac{f(w)}{w-z}$ is continuous, the integrals over the sides of the corridor cancel out

$$0 = \int_{\gamma} \frac{f(w)}{w-z} - \int_{\partial B(z,\varepsilon)} \frac{f(w)}{w-z}\,,$$ enough to prove: $f(z) = \frac{1}{2\pi i} \int_{\partial B(z,\varepsilon)} \frac{f(w)}{w-z} dw$

since f is holomorphic at z: $\frac{f(w)-f(z)}{w-z} \to f'(z_0)$ as $w \to z$

in particular, it is bounded: $\left|\frac{f(w)-f(z)}{w-z}\right| \le M$ on a disc

$$\int_{\partial B(z,\varepsilon)} \frac{f(w)}{w-z} = \underbrace{\int_{\partial B(z,\varepsilon)} \frac{f(w)-f(z)}{w-z}}_{} + f(z) \cdot \int_{\partial B(z,\varepsilon)} \frac{1}{w-z}\, dw$$

$| \ | \le M \cdot \text{length } \partial B(z,\varepsilon) \to 0$ as $\varepsilon \to 0$

$$f(z) \cdot \int_{\partial B(z,\varepsilon)} \frac{1}{w-z}\, dw = f(z) \cdot \int_0^{2\pi} \frac{(\varepsilon e^{it})'}{\varepsilon e^{it}}\, dt = f(z) \cdot \int_0^{2\pi} i = 2\pi i f(z$$

) i) $$\int_{\gamma} \frac{z^5+i}{z-i} = 2\pi i\,(i^5+i) = 2\pi i\, 2i = -4\pi$$

ii) $f(z) = e^{z^3}$ $\quad f''(1) = \cancel{}\,15\,e$

$$\int_{\gamma} \frac{f(z)}{(z-1)^3} = \frac{2\pi i}{2!}\, \cancel{}\,15e = 15\pi i e$$

a) i) z_0 is an isolated singularity, f can be extended to z_0 s.t. it becomes a holomorphic function also at z_0.

ii) $$f(z) = \frac{a_{-m}}{(z-z_0)^m} + \dots + \frac{a_{-1}}{z-z_0} + \text{holomorphic function} \qquad a_{-m} \neq 0$$

on a neighbourhood of z_0

iii) an isolated singularity which is neither removable nor pole

b) i) $z + \frac{1}{z} \to \infty$ as $z \to 0$

$f(z) \to 0$ as $z \to 0$, removable

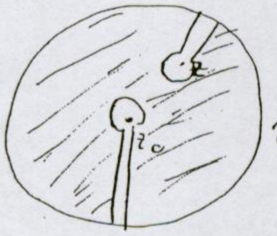
ii) if z is positive and real: $z^2 e^{1/z} \to \infty$ as $z \to 0$

if $z = ix$, $x \in \mathbb{R}$, $x \to 0$: $z^2 e^{1/z} \to 0$ as $z \to 0$

essential

iii) $\frac{1}{f(z)} \to 0$ as $z \to 0$ pole

c)



γ_ε = curve with 2 keyholes

ε = radius of circles around z, z_0

$$\frac{1}{2\pi i}\int_{\gamma_\varepsilon} \frac{f(w)}{w-z}\, dw = 0$$

as width of corridors $\to 0$:

$$0 = \frac{1}{2\pi i}\int_{\gamma} \frac{f(w)}{w-z}\, dw - \underbrace{\frac{1}{2\pi i}\int_{\partial B(z_0,\varepsilon)} \frac{f(w)}{w-z}\, dw}_{|\;| \le C\cdot\varepsilon} - \underbrace{\frac{1}{2\pi i}\int_{\partial B(z,\varepsilon)} \frac{f(w)}{w-z}\, dw}_{f(z)}$$

define $f(z_0) = \frac{1}{2\pi i}\int_{\gamma} \frac{f(w)}{w-z_0}\, dw$

then $f(z) = \frac{1}{2\pi i}\int_{\gamma} \frac{f(w)}{w-z}\, dw$ everywhere, which is holomorphic on the whole disc

⑤ d) if f has an essential singularity at z_0

then $f(B(z_0, r) \setminus \{z_0\})$ is dense in $\mathbb{C}$

for any $r > 0$ for which $B(z_0, r) \subseteq$ domain of f.

a) $f(z) = \dfrac{1}{(z^2+1)^2(z^2+4)} = \dfrac{1}{(z+i)^2(z-i)^2(z+2i)(z-2i)}$

poles: $\pm i$ multiplicity 2

$\pm 2i$ multiplicity 1

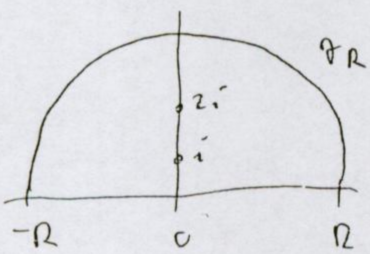
b) $\operatorname{Res}_{2i} f = \dfrac{1}{(2i+i)^2(2i-i)^2(2i+2i)} = \dfrac{1}{(3i)^2 i^2 4i} = \dfrac{-i}{36}$

$\operatorname{Res}_i f = \left(\dfrac{1}{(z+i)^2(z^2+4)}\right)'(i)$

$\left(\dfrac{1}{(z+i)^2(z^2+4)}\right)' = \dfrac{-2(z+i)(z^2+4) - (z+i)^2 2z}{(z+i)^4(z^2+4)^2}$

$= \dfrac{-2(z^2+4) - (z+i)2z}{(z+i)^3(z^2+4)^2}$

$\operatorname{Res}_i f = \dfrac{-2(i^2+4) - 2i\cdot 2i}{(2i)^3(i^2+4)^2} = \dfrac{-2\cdot 3 + 2\cdot 2}{-8i\cdot 3^2} = \dfrac{-2}{-72i} = \dfrac{-i}{36}$



$R > 2$

on γ_R: $\dfrac{1}{z^2+1} \leq \dfrac{1}{R^2-1}$

$\dfrac{1}{z^2+4} \leq \dfrac{1}{R^2-4}$

$\left| \int_{\gamma_R} f(z)\,dz \right| \leq \int_0^{\pi} \dfrac{1}{(R^2-1)^2(R^2-4)} \cdot R = O\left(\dfrac{1}{R^5}\right) \to 0$

Hence $\int_{-\infty}^{\infty} f(x)\,dx = \lim_{R\to\infty} \int_{-R}^{R} f(x)\,dx = 2\pi i \left(\dfrac{-i}{36} + \dfrac{-i}{36}\right) = \dfrac{\pi}{9}$